Bosonic Negative Energy Enhancement

Senior Honors Thesis Defense

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May 2, 2024

1 Introduction

2 Clarifications

3 Symplectic and Numerical Methods

4 Results?

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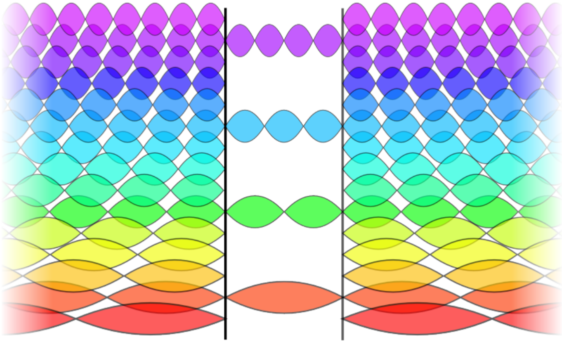
Casimir Effect

- Consider 2 nearby *grounded* parallel plates:



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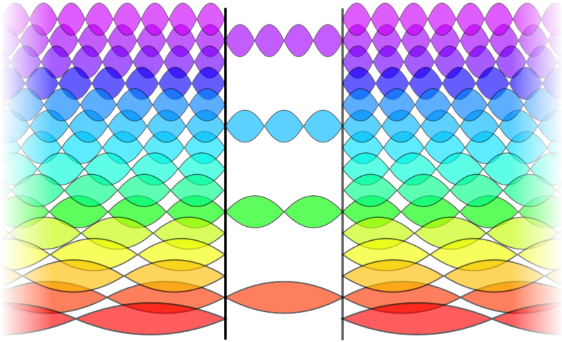

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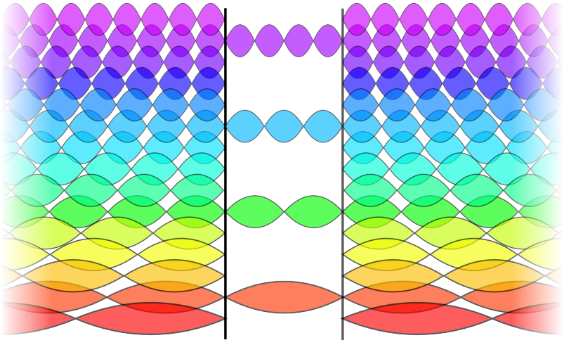

- There is a net force that pushes the plates towards each other
  - This also occurs if we remove the plates and impose periodic boundary conditions

Regularization

- We know that Quantum Field Theory (QFT) is riddled with infinities (throughout we set $c = \hbar = 1$):

$$\mathcal{L} = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 - \frac{m^2}{2}\varphi^2 \implies E_{gs}^{FT} = \sum_{n=1}^{\infty}\sqrt{\frac{n^2}{R^2} + m^2},$$

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$$E_{gs}^{FT}(m=0) = -\frac{1}{12R}, \quad E_{gs}^{FT}(m) = -\frac{1}{12R}f(mR)$$

- We have found *negative energy* in our system, which pushes the plates together

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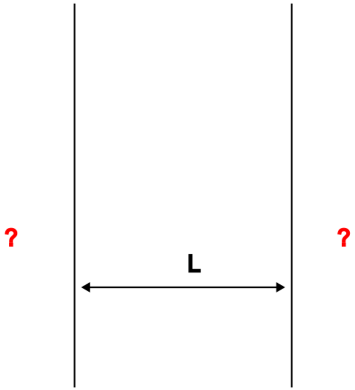

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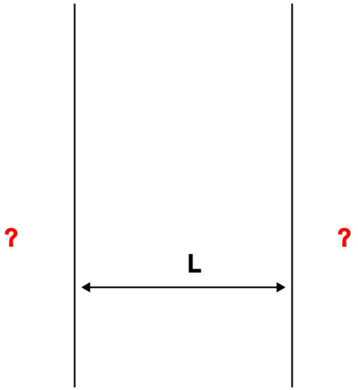

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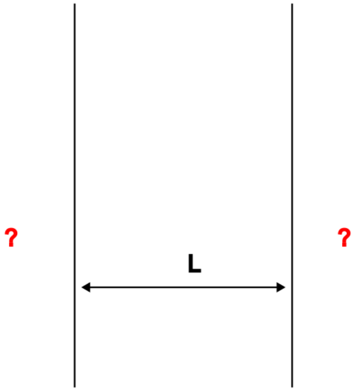

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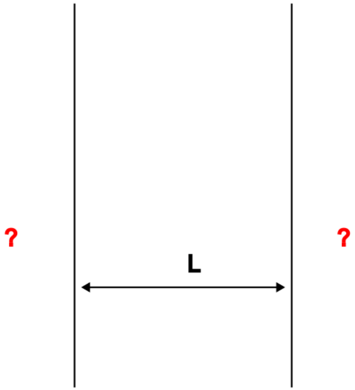
- But, are there states with lower uniform negative energies?



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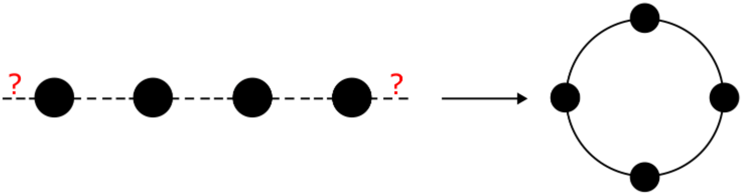
What / why are PBC?

- Imagine living on a circle:



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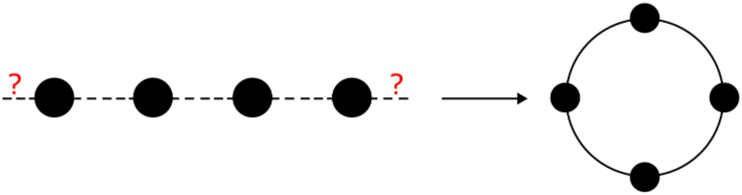

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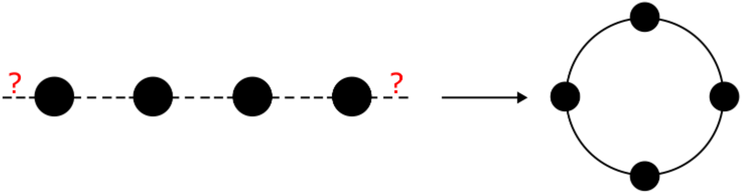

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- PBC are natural in the context of bosons (photons)

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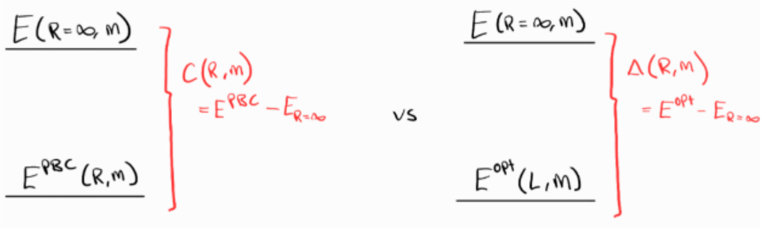

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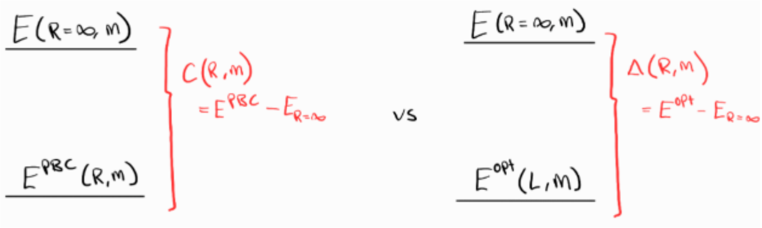

- We consider our optimized energy to be *negative* if $E^{opt} < E_{R\to\infty}$, and to be *enhanced negative energy* if $E^{opt} < E^{PBC}$

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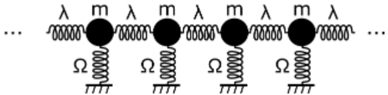
Why Symplectic? pt 1

- We UV regulate our field theory with a lattice of N coupled harmonic oscillators:

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2\mu} + \frac{1}{2}\mu\Omega^2 q_i^2 + \frac{\lambda}{2}\left((q_i - q_{i-1})^2 + (q_i - q_{i+1})^2 \right) \right]$$



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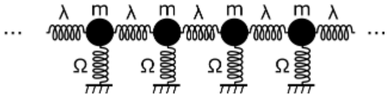

- For $N = 1$, this is:

$$H = \frac{1}{2} \begin{matrix} & q & p \\ q & B & 0 \\ p & 0 & A \end{matrix}$$

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$$H \to \tilde{H} = \frac{\sqrt{AB}}{2} \begin{array}{c} \\ q \\ p \end{array} \begin{array}{c} \begin{array}{cc} q & p \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

- This matrix form shows up in a fundamental theorem of symplectic transformations: *Williamson's theorem*

Williamson's Theorem

- **Theorem:** For a positive, symmetric matrix, M, there exists a symplectic transformation S such that:

$$SMS^T = \bigoplus_i d_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $d_i > 0$ are known as symplectic eigenvalues, and $SJS^T = J$, for $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

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- For the $N = 1$ case above, $d_i = \frac{\sqrt{AB}}{2}$
- This transformation sends our original Hamiltonian (in this case, M) to one describing N uncoupled oscillators

Gradient Descent

- We can write our density matrix (the state of our system) as:

$$\rho = \frac{1}{Z} e^{-\sum_a \beta_a H_a},$$

$$H_a = \frac{p_a^2}{2\mu} + \frac{1}{2}\mu\Omega^2 q_a^2 + \frac{\lambda}{2}\left((q_a - q_{a-1})^2 + (q_a - q_{a+1})^2\right)$$

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- We use ρ and H_a to calculate the energy expectation values:

$$E_a = \langle H_a \rangle = \sum_i h_1(d_i) \frac{\mathrm{d}d_i}{\mathrm{d}\beta_a},$$

where $\frac{\mathrm{d}d_i}{\mathrm{d}\beta_a}$ is the first order *perturbative* contribution to the ith symplectic eigenvalue, d_i (more on this later)

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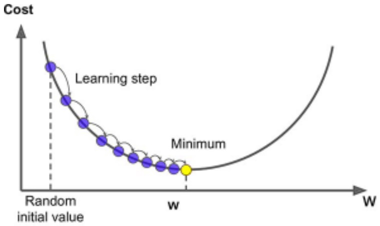

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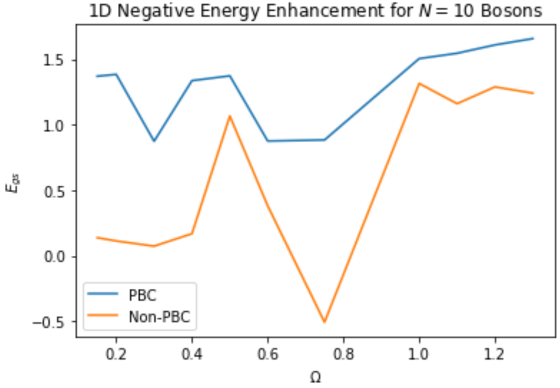

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  - Technically, this was like a harder version of the usual QM perturbation theory

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- The second derivative of d_i is:

$$D^{(2)} = D_0 C^T + PB^T + BD_0 B^T + BP + CD_0,$$

which we are still implementing numerically. B and C are matrices that show up in our symplectic perturbation theory

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- Once we implement this second derivative formula, we can easily study the problem at hand

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- This will give us a scaling relationship between the two. For certain values of this scaling, we expect that in $3+1$D we will have an unboundedly negative ground state energy

Thank You